§13.4 Theory of Curves

- We set out to describe the acceleration vector $\vec{a} = \frac{d}{dt}\vec{v}(t)$. We have:

$$\vec{a} = \frac{d}{dt}\left(\|\vec{v}\|\vec{T}\right) = \underbrace{\left(\frac{d}{dt}\|\vec{v}\|\right)}_{\frac{d^2s}{dt^2}}\vec{T} + \underbrace{\|\vec{v}\|}_{\frac{ds}{dt}}\underbrace{\left(\frac{d\vec{T}}{dt}\right)}_{\perp \vec{v}}$$

[Recall: $\|\vec{T}(t)\| = 1 \Rightarrow 1 = \vec{T}(t)\cdot\vec{T}(t) = \|\vec{T}(t)\|^2$

$\Rightarrow 0 = \vec{T}'\cdot\vec{T} + \vec{T}\cdot\vec{T}' = 2\vec{T}'\cdot\vec{T}$ ✓]

Theorem: If $\frac{d\vec{T}}{dt} \neq 0$, then $\frac{d\vec{T}}{dt} \perp \vec{T}$ so

$$\frac{d\vec{T}}{dt} = \left\|\frac{d\vec{T}}{dt}\right\|\vec{N}$$

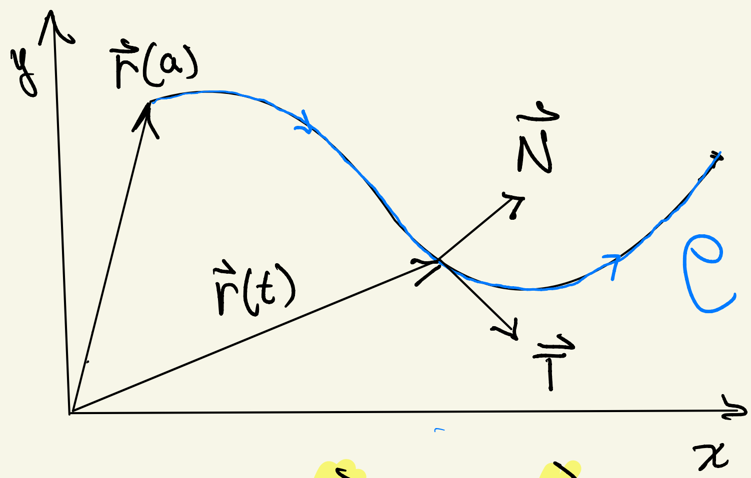
where

$$\vec{N} \equiv \vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} \equiv \frac{d\vec{T}/dt}{\|d\vec{T}/dt\|}$$

is the Principle Normal Vector

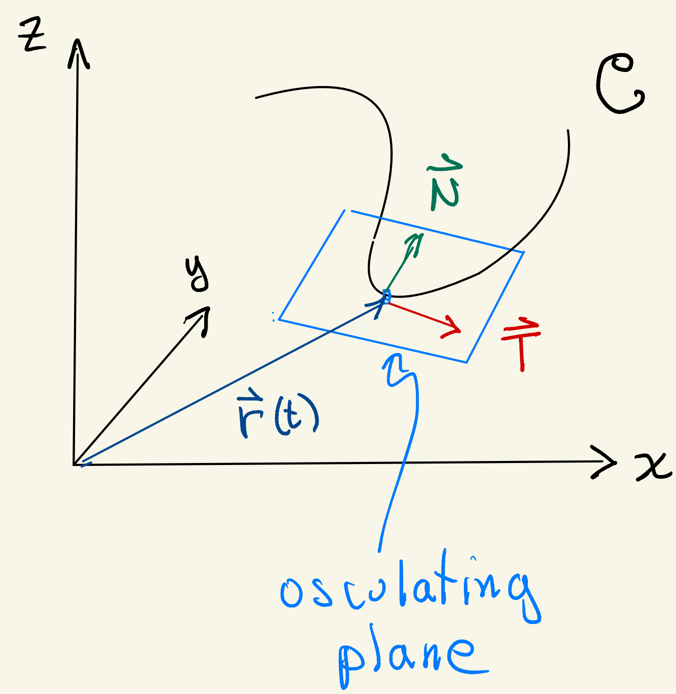
- In $\mathbb{R}^2$: ($\vec{r}(t) = (x(t), y(t))$)

the Principal Normal $\vec{N}$ points orthogonal to $\vec{T}$ in the direction $\mathcal{C}$ is curving.



- In $\mathbb{R}^3$, the plane spanned by $\vec{T}$ and $\vec{N}$ is the osculating plane, the plane in which the curve most closely lies @ $\vec{r}(t)$

Picture: The Principal Unit Normal $\vec{N}$ gives the direction and plane into which $\mathcal{C}$ is "curving away from $\vec{T}$"



$$\vec{N} = \frac{\left(\frac{d\vec{T}}{dt}\right)}{\left\|\frac{d\vec{T}}{dt}\right\|}$$

- Putting it all together:

$$\vec{a} = \frac{d}{dt}\left(\|\vec{v}\|\vec{T}\right) = \underbrace{\frac{d}{dt}\|\vec{v}\|}_{\frac{d}{dt}\frac{ds}{dt}}\vec{T} + \underbrace{\|\vec{v}\|\frac{d\vec{T}}{dt}}_{\frac{ds}{dt}\left\|\frac{d\vec{T}}{dt}\right\|\vec{N}}$$

So

$$\boxed{\vec{a} = \underbrace{\frac{d^2s}{dt^2}}_{\text{the scalar acceleration}}\vec{T} + \underbrace{\frac{ds}{dt}}_{\text{speed } \frac{ds}{dt} = v}\underbrace{\left\|\frac{d\vec{T}}{dt}\right\|}\vec{N}}$$

a measure of the "curvature" at $\vec{r} = \vec{r}(t)$ ...

$\kappa = \left\|\frac{d\vec{T}}{dt}\right\|$ is called the Curvature

Said differently:

$$\boxed{\vec{a} = a_T\vec{T} + a_N\vec{N}}$$

$$a_T = \frac{d^2s}{dt^2} \qquad a_N = v\left\|\frac{d\vec{T}}{dt}\right\|$$

Eg: $\vec{a}\cdot\vec{T} = \left(a_T\vec{T} + a_N\vec{N}\right)\cdot\vec{T}$

$$= a_T\underbrace{\vec{T}\cdot\vec{T}}_{1} + a_N\underbrace{\vec{N}\cdot\vec{T}}_{0} = a_T = \underbrace{\frac{d^2s}{dt^2}}_{\text{scalar acceleration}}$$

Summary: $\vec{a} = a_T \vec{T} + a_N \vec{N}$

a_T: component of $\vec{a}$ in direction $\vec{T}$

a_N: component of $\vec{a}$ in direction $\vec{N}$

where $a_T = \dfrac{dv}{dt}$, $a_N = v \left\| \dfrac{d\vec{T}}{dt} \right\|$ $\qquad v = \dfrac{ds}{dt}$

So we have proven:

Theorem: $\vec{a} \cdot \vec{T} = \dfrac{dv}{dt}$, $\vec{a} \cdot \vec{N} = v \left\| \dfrac{d\vec{T}}{dt} \right\|$

- It remains to understand $\left\| \dfrac{dT}{dt} \right\|$

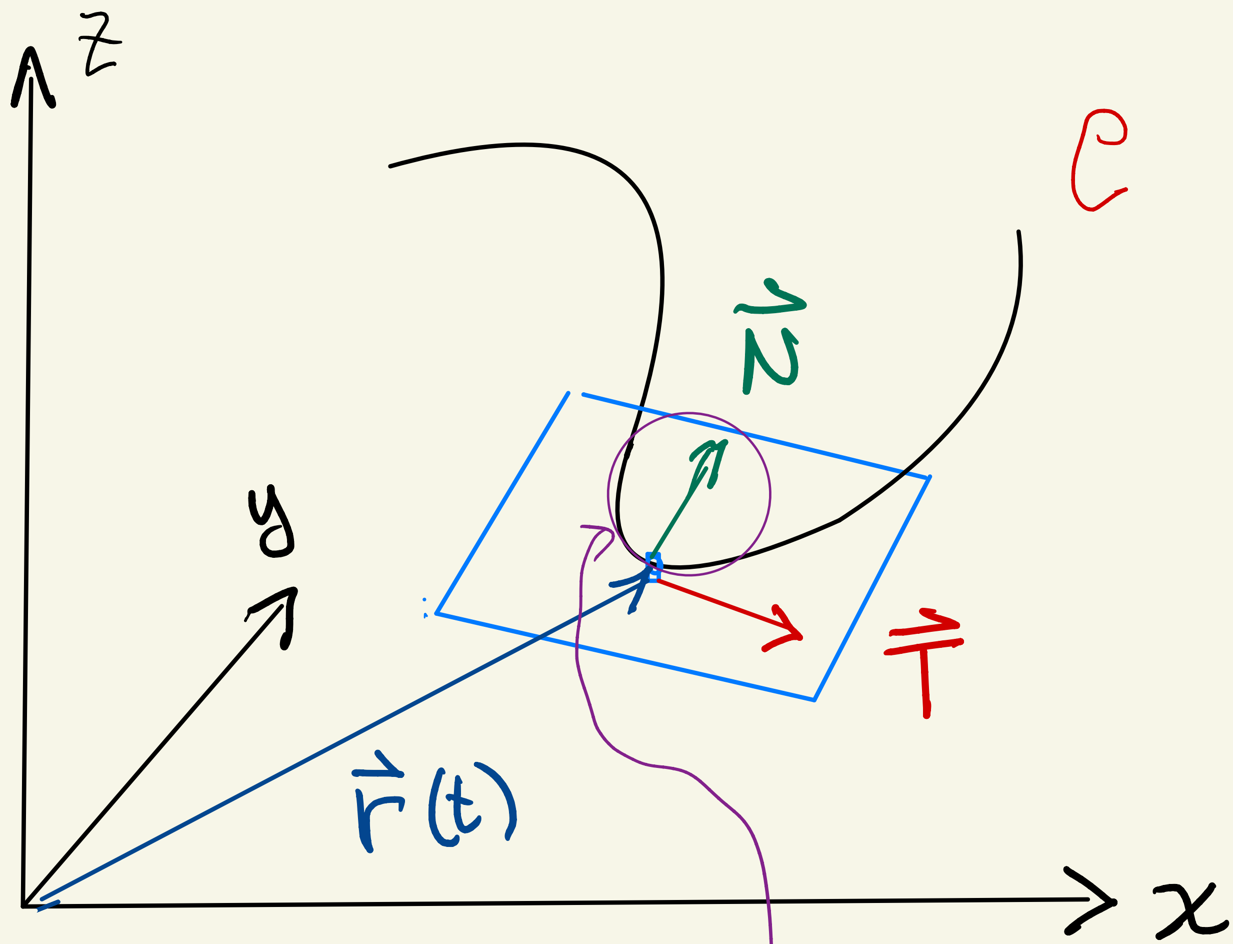
Theorem: $\left\| \dfrac{dT}{dt} \right\| = \kappa v$ $\qquad \left(v = \dfrac{ds}{dt} \right)$

where $\kappa = \dfrac{1}{r} = \dfrac{1}{\text{radius of curvature}}$

r = "radius of the circle that best fits the curve at point $\vec{r}(t)$"

Defn: $\kappa = \kappa(t) = $ curvature of $\mathcal{C}$ at $\vec{r}(t)$

Picture



"osculating circle"
lies in the osculating plane
r = radius of curvature

$$\kappa = \frac{1}{r}$$

Conclude: Geometrical Interpretation of the acceleration vector:

$$\vec{a} = a_T \vec{T} + a_N \vec{N} \qquad v = \frac{ds}{dt}$$

$$= \frac{d^2s}{dt^2}\vec{T} + \kappa v^2 \vec{N}$$

$a_T = \frac{d^2s}{dt^2}$ is the scalar acceleration

r = radius of curvature

$$a_N = v^2 \kappa = \frac{v^2}{r}$$

v = velocity

$$\vec{T} = \frac{\vec{v}}{\|\vec{v}\|} \qquad \vec{N} = \frac{\frac{d\vec{T}}{dt}}{\left\|\frac{d\vec{T}}{dt}\right\|} \qquad \left(\text{or } \vec{N} = 0 \text{ if } \frac{d\vec{T}}{dt} = 0\right)$$

(assume $\vec{v} \neq 0$)

this is the theory – we now do some examples –

Example 1: Show that $\kappa = \left\| \frac{d\vec{T}}{ds} \right\|$

Soln: If we are given $\vec{T}(t)$, then

$$\frac{d\vec{T}}{dt} = \underbrace{\left\| \frac{d\vec{T}}{dt} \right\|}_{\text{length}} \underbrace{\vec{N}}_{\text{direction (unit)}} = v\kappa\vec{N}$$

But
$$\frac{d\vec{T}}{ds} = \frac{d}{ds}\vec{T}(t(s)) = \frac{d\vec{T}}{dt} \cdot \underbrace{\frac{dt}{ds}}_{1/v}$$

$$= \cancel{v}\kappa\vec{N}\frac{1}{\cancel{v}} = \kappa\vec{N}$$

Therefore $\left\| \frac{d\vec{T}}{ds} \right\| = \|\kappa\vec{N}\| = \kappa$ ✓

Example 2

Let $\vec{r}(t) = t\,\underset{\sim}{i} + \frac{1}{2}t^2\,\underset{\sim}{j}$

Find: $\vec{v}$, $\vec{a}$, $\frac{ds}{dt}$, $\vec{T}$, $\frac{d^2s}{dt^2}$, a_T, $\vec{N}$, a_N, κ

Soln (a) $\vec{v} = \frac{d\vec{r}}{dt} = \underset{\sim}{i} + t\,\underset{\sim}{j} = \overrightarrow{(1, t)}$

(b) $\vec{a} = \frac{d\vec{v}}{dt} = 0\,\underset{\sim}{i} + \underset{\sim}{j} = \underset{\sim}{j} = \overrightarrow{(0, 1)}$

(c) $\frac{ds}{dt} = v = \|\vec{v}\| = \sqrt{1+t^2}$

(d) $\vec{T} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\underset{\sim}{i} + t\,\underset{\sim}{j}}{\sqrt{1+t^2}} = \frac{1}{\sqrt{1+t^2}}\underset{\sim}{i} + \frac{t}{\sqrt{1+t^2}}\underset{\sim}{j}$

(e) $\frac{d^2s}{dt^2} = \vec{a} \cdot \vec{T} = \overrightarrow{(0, 1)} \cdot \overrightarrow{\left(\frac{1}{\sqrt{1+t^2}}, \frac{t}{\sqrt{1+t^2}}\right)} = \frac{t}{\sqrt{1+t^2}}$

(f) $a_T = \vec{a} \cdot \vec{T} = \frac{d^2s}{dt^2} = \frac{t}{\sqrt{1+t^2}}$

(g) $\vec{N} = \dfrac{1}{\left\| \dfrac{d\vec{T}}{dt} \right\|} \dfrac{d\vec{T}}{dt}$, $\dfrac{d\vec{T}}{dt} = \dfrac{d}{dt}\overrightarrow{\left((1+t^2)^{-1/2}, \dfrac{t}{\sqrt{1+t^2}} \right)}$

$$\frac{d\vec{T}}{dt} = -\frac{1}{\cancel{2}}(1+t^2)^{-3/2}\cancel{2}t\,\underset{\sim}{i} + \frac{\sqrt{1+t^2}\cdot 1 - t\frac{1}{\cancel{2}}(1+t^2)^{-1/2}\cancel{2}t}{1+t^2}\cdot\underset{\sim}{j}$$

$$= \frac{-t}{(1+t^2)^{3/2}}\underset{\sim}{i} + \frac{(1+t^2)-t^2}{(1+t^2)^{3/2}}\underset{\sim}{j} = \frac{1}{(1+t^2)^{3/2}}\overrightarrow{(-t,1)}$$

$$\left\| \frac{d\vec{T}}{dt} \right\| = \frac{1}{(1+t^2)^{3/2}}\left\|\overrightarrow{(-t,1)}\right\| = \frac{\sqrt{1+t^2}}{(1+t^2)^{3/2}} = \frac{1}{1+t^2}$$

Thus! $\vec{N} = \underbrace{(1+t^2)}_{\frac{1}{\left\|\frac{d\vec{T}}{dt}\right\|}}\underbrace{\dfrac{1}{(1+t^2)^{3/2}}\overrightarrow{(t,1)}}_{\frac{d\vec{T}}{dt}} = \dfrac{1}{\sqrt{1+t^2}}\overrightarrow{(t,1)}$

Check: $\|\vec{N}\| = \dfrac{1}{\sqrt{1+t^2}}\left\|\overrightarrow{(t,1)}\right\| = 1$ ✓

(h) $a_N = \vec{a}\cdot\vec{N} = \underbrace{\overrightarrow{(0,1)}}_{\text{vector}}\cdot\underbrace{\left(\dfrac{1}{\sqrt{1+t^2}}\right)}_{\text{scalar}}\underbrace{\overrightarrow{(t,1)}}_{\text{vector}} = \dfrac{1}{\sqrt{1+t^2}}$

(i) $\kappa v^2 = a_N$ so $\kappa = \dfrac{a_N}{v^2} = \dfrac{1}{\sqrt{1+t^2}}\cdot\left(\sqrt{1+t^2}\right)^2 = \sqrt{1+t^2}$

Example ③ Show that when $a_N = 0$, $v \neq 0$, motion is along a straight line —

Soln: $\vec{a} = \dfrac{d^2 s}{dt^2}\vec{T} + \underbrace{v^2 \kappa \vec{N}}_{a_N}$

Thus if $a_N = 0$ either $\cancel{v = 0}$ or $\kappa = 0$

But $\kappa = \dfrac{dT}{ds} = 0 \Rightarrow T = \text{const}$

I.e. $\dfrac{d\vec{T}}{ds} = \dfrac{d}{ds}\left(x(s)\,\underset{\sim}{i} + y(s)\,\underset{\sim}{j} + z(s)\,\underset{\sim}{k}\right) = 0$

$x = \text{const}$ $y = \text{const}$ $z = \text{const}$ $\Rightarrow \vec{T} = \text{const}$

$\therefore$ $\vec{r}(s) = \underbrace{\vec{T}}_{\text{const}} \cdot s + \underbrace{\vec{r}_0}_{\text{const}}$

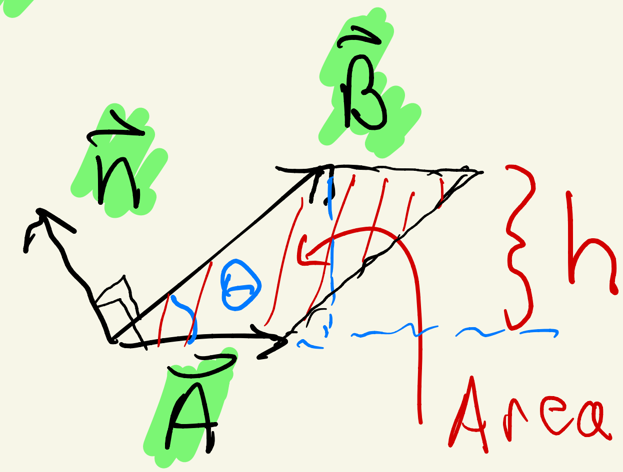
straight line

Example 4 Find a formula for κ in terms of $\vec{v}$ and $\vec{a}$

Soln: $$\vec{a} = a_T \vec{T} + a_N \vec{N}$$

Recall cross product:

$$\vec{A} \times \vec{B} = \|\vec{A}\|\|\vec{B}\| \sin\theta \; \vec{n}$$



Area = $\|\vec{A}\| \underbrace{\|\vec{B}\| \sin\theta}_{h}$

$$\|\vec{v} \times \vec{a}\| = \|\vec{v} \times (a_T \vec{T} + a_N \vec{N})\|$$

$$= \|a_T \underbrace{\cancel{\vec{v} \times \vec{T}}}_{0} + a_N \vec{v} \times \vec{N}\|$$

$$= a_N \|\vec{v} \times \vec{N}\| = \kappa v^2 \|\vec{v} \times \vec{N}\|$$

$$= \kappa v^3 \underbrace{\cancel{\|\vec{T} \times \vec{N}\|}}_{1}$$

So $$\boxed{\kappa = \frac{\|\vec{v} \times \vec{a}\|}{v^3}}$$

Example ④ Find the equation for the osculating plane at $\vec{r}(2)$ for the helix

$$\vec{r}(t) = 3\cos t\,\underset{\sim}{i} + 3\sin t\,\underset{\sim}{j} + t\,\underset{\sim}{k}$$

Soln: $\vec{T}(t) = \dfrac{\vec{v}(t)}{\|\vec{v}(t)\|} = \dfrac{-3\sin t\,\underset{\sim}{i} + 3\cos t\,\underset{\sim}{j} + \underset{\sim}{k}}{\|\vec{v}(t)\|}$

(Idea)

$$\|\vec{v}(t)\| = \sqrt{9\sin^2 t + 9\cos^2 t + 1} = \sqrt{10}$$

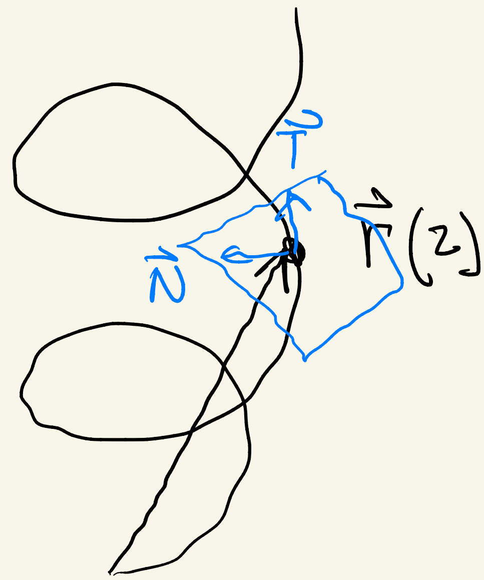
$$\vec{T}(t) = \frac{\overrightarrow{(-3\sin t, 3\cos t, 1)}}{\sqrt{10}}$$

$$\frac{d\vec{T}}{dt} = \frac{1}{\sqrt{10}}\overrightarrow{(-3\cos t, -3\sin t, 0)}$$

$$\vec{N} = \overrightarrow{(-\cos t, -\sin t, 0)}$$

Osculating Plane is the $\vec{r}(2)$ plus the span of $\vec{T}$ & $\vec{N}$

Equation of plane thru P_0



$$\overrightarrow{P_0P} \cdot \vec{n} = 0$$

$\vec{n} = \vec{T} \times \vec{N}$, $P_0 = \vec{r}(2)$, $P = (x, y, z)$

$$\vec{V} \times \vec{N} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -3\sin t & 3\cos t & 1 \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

$$= \underset{\sim}{i}(-\sin t) - \underset{\sim}{j}(-\cos t) + (+3\sin^2 t + 3\cos^2 t)\underset{\sim}{k}$$

$$= -\sin t\, \underset{\sim}{i} + \cos t\, \underset{\sim}{j} + 3\underset{\sim}{k}$$

Example (5): Show that for uniform motion on a circle of radius r, the curvature $\kappa = \frac{1}{r}$

Soln:

$$\vec{r}(t) = \overrightarrow{(x_0, y_0)} + r\overrightarrow{(\cos t, \sin t)}$$

$$\vec{v}(t) = r\overrightarrow{(-\sin t, \cos t)}, \quad v = r$$

$$\vec{a}(t) = r\underbrace{\overrightarrow{(-\cos t, -\sin t)}}_{\vec{N}}$$

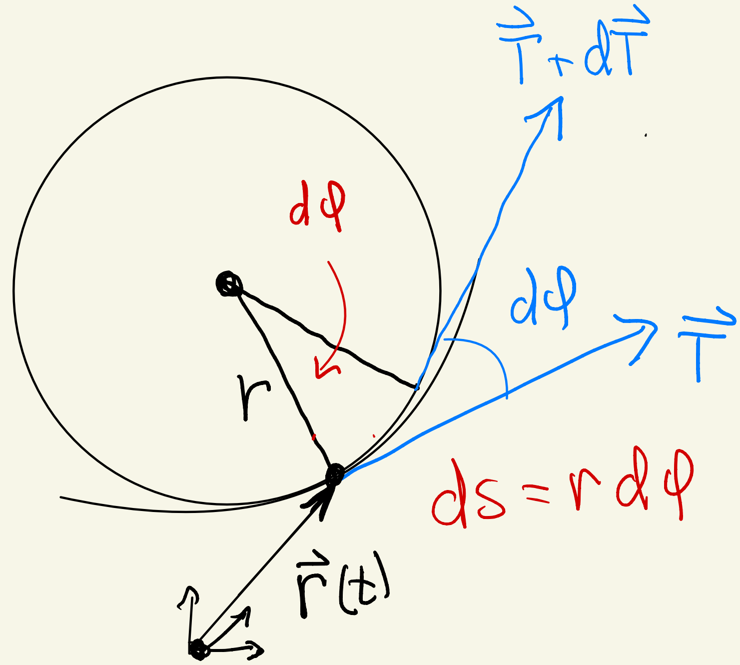
$$a_N = \vec{a} \cdot \vec{N} = r$$

In general: $a_N = \kappa v^2 = \kappa r^2$

Thus $r = \kappa r^2 \Rightarrow \kappa = \frac{1}{r}$ ✓

Q: Why is $\left\| \frac{d\vec{T}}{ds} \right\| = \kappa = \frac{1}{r}$ in general?

Soln: Restrict to osculating plane –



Then a small motion away from $\vec{r}(t)$ gives

$$ds = r\,d\varphi$$

$$\vec{T} = \cos\varphi\,\underset{\sim}{i} + \sin t\,\underset{\sim}{j}$$

$$\underbrace{\left\| \frac{d\vec{T}}{ds} \right\|}_{\kappa} = \left\| \underbrace{\frac{d\vec{T}}{d\varphi}}_{\substack{\vec{N} \\ \text{unit}}} \underbrace{\frac{d\varphi}{ds}}_{1/r} \right\| = \left| \frac{d\varphi}{ds} \right| = \frac{1}{r}$$

$$\boxed{\kappa = \frac{1}{r}}$$

General Theory of Curves:

$$\vec{T} = \frac{\vec{V}}{\|\vec{V}\|}, \quad \vec{N} = \frac{dT/ds}{\|dT/ds\|} = \frac{1}{\kappa}\frac{dT}{ds}$$

Define Binormal $\vec{B} = \vec{T} \times \vec{N}$

Get:

$$\frac{d\vec{T}}{ds} = \kappa \vec{N}$$

$$\frac{d\vec{N}}{ds} = -\kappa \vec{T} + \tau \vec{B}$$

$$\frac{d\vec{B}}{ds} = -\tau N$$

$\kappa \equiv \kappa(s) =$ curvature

$\tau \equiv \tau(s) =$ torsion

Matrix Form – Equations for $(\vec{T}(s), \vec{N}(s), \vec{B}(s))$

Frenet-Serret Equations

F – 1847

S – 1851

$$\begin{pmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{pmatrix}'(s) = \underbrace{\begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}}_{\text{anti-symetric}} \begin{pmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{pmatrix}$$

Theorem: Everything about $\mathcal{C}$ is determined by curvature $\kappa(s)$ & torsion $\tau(s)$